ELSEVIER
Discrete Mathematics 241 (2001) 387–393
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

A note on equal unions in families of sets

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Abstract

A family of sets has the *equal union property* if and only if there exist two nonempty disjoint subfamilies having the same union. We prove that any n nonempty subsets of an n-element set have the equal union property if the sum of their cardinalities exceeds $n(n+1)/2$. This bound is tight. Among families in which the sum of the cardinalities equals $n(n+1)/2$, we characterize those having the equal union property. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Equal union property; Sign-nonsingular

1. Introduction

Let $F=(S_1,\ldots,S_m)$ be a finite, nonempty family of nonempty subsets of a finite set X. We say that F has the *equal union property* if and only if there exist nonempty disjoint sets $\Gamma,\Delta\subseteq\{1,\ldots,m\}$, for which

$$\bigcup_{\gamma\in\Gamma} S_\gamma=\bigcup_{\delta\in\Delta} S_\delta. \tag{1}$$

We do not require that the S_i be distinct, but note that if $S_i=S_j$ for $i\neq j$, then trivially (1) holds by taking $\Gamma=\{i\}$ and $\Delta=\{j\}$.

Lindström appears to have been the first to consider the equal union property; using combinatorial methods he gave a sufficient condition for r equal unions in [5].

In [12] Tverberg used his celebrated extension of Radon's convexity theorem to give an algebraic proof of Lindström's results.

Let $F=(S_1,\ldots,S_m)$ where each $S_i\subseteq X=\{x_1,\ldots,x_n\}$. The *incidence matrix* of F is the $n\times m$ 0–1 matrix $M=(m_{ij})$ in which $m_{ij}=1$ if and only if $x_i\in S_j$. For each $x\in X$, its *degree* is the number of sets containing it. Note that the number of 1's in column j is the cardinality of S_j, and the number of 1's in row i is the degree of x_i.

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PII: S0012-365X(01)00124-8

For an $n\times m$ real matrix $A=(a_{ij})$, let $Q(A)$ denote the set of all $n\times m$ matrices having the same *sign pattern* as A. That is, $B=(b_{ij})$ is in $Q(A)$ if $b_{ij}=0$ if and only if $a_{ij}=0$, and otherwise b_{ij} and a_{ij} have the same sign. We say that A is an *L-matrix* if and only if for every $B\in Q(A)$, $Bx=0$ implies $x=0$, or equivalently, the columns of every $B\in Q(A)$ are linearly independent. Square L-matrices are called *sign-nonsingular* [6].

The following theorem and its proof are inspired by techniques introduced by Tverberg [12]. Similar ideas appear in [4], but without reference to the equal union property.

Theorem 1. *A sequence $F=(S_1,\ldots,S_m)$ of nonempty subsets of $X=\{x_1,\ldots,x_n\}$ has the equal union property if and only if its incidence matrix M is not an L-matrix.*

Proof. Assume F has the equal union property, and let Γ and Δ be the index sets in (1), and let $\bigcup_{\gamma\in\Gamma} S_\gamma=\bigcup_{\delta\in\Delta} S_\delta=U$. Let u be the $n\times 1$ characteristic vector of U. And let v and w be the $m\times 1$ characteristic vectors of Γ and Δ, respectively. For each $x\in X$ define

$$\deg_\Gamma(x)=|\{\gamma\in\Gamma\mid x\in S_\gamma\}|,$$
$$\deg_\Delta(x)=|\{\delta\in\Delta\mid x\in S_\delta\}|.$$

Note that these degrees are both zero when $x\notin U$, and are both positive (though possibly different) when $x\in U$. Now define $B=(b_{ij})$ in $Q(M)$ as follows:

$$b_{ij}=\begin{cases}\frac{1}{\deg_\Gamma(x_i)} & \text{if } x_i\in S_j \text{ and } j\in\Gamma,\\ \frac{1}{\deg_\Delta(x_i)} & \text{if } x_i\in S_j \text{ and } j\in\Delta.\end{cases}$$

Let B agree with M in all other positions. It follows that $Bv=u$ and $Bw=u$. Therefore $B(v-w)=0$, where $v\neq w$, so M is not an L-matrix.

Conversely, assume M is not an L-matrix, and $Bz=0$ for some $B\in Q(M)$ and $z\neq 0$. Define v and w as follows:

$$v_i=\begin{cases} z_i & \text{if } z_i>0,\\ 0 & \text{otherwise,}\end{cases}$$

$$w_i=\begin{cases} -z_i & \text{if } z_i<0,\\ 0 & \text{otherwise.}\end{cases}$$

Then v and w are each nonnegative vectors with $z=v-w$. Next let $\Gamma=\{i\mid z_i>0\}$, and $\Delta=\{i\mid z_i<0\}$. Since each $S_i\neq\emptyset$, the columns of M, and of B, are nonzero. Since B has no negative entries or zero columns, $Bz=0$ and $z\neq 0$ imply that z must have both a positive and negative component. Therefore both v and w are nonzero, and so both Δ and Γ are nonempty. Since B and v are nonnegative, the vector Bv will pick out those rows of B for which some column indexed by Γ has a nonzero entry in that row. Since the columns of B have the same sign pattern as the characteristic vectors of the sets S_i, Bv will be nonzero exactly in those rows corresponding to $\bigcup_{\gamma\in\Gamma} S_\gamma$. Similarly, Bw will be nonzero in exactly those rows corresponding to $\bigcup_{\delta\in\Delta} S_\delta$. Since $Bv = Bw$ we must have $\bigcup_{\gamma\in\Gamma} S_\gamma = \bigcup_{\delta\in\Delta} S_\delta$. □

Two immediate consequences to Theorem 1 are

Corollary 1. *Let F be a family of n nonempty subsets of an n-element set, having $n \times n$ incidence matrix M. Then F has the equal union property if and only if M is not sign-nonsingular.*

Corollary 2. *Any sequence of at least $k+1$ nonempty sets whose union has k elements has the equal union property.*

Proof. Let M be the family's incidence matrix. Since M has more columns than rows, its columns are linearly dependent. Hence there is a nonzero x for which $Mx = 0$. This implies M is not an L-matrix. By Theorem 1, the sequence must have the equal union property. □

The family $\{\emptyset, \{1\}\}$ shows that it is necessary in Corollary 2 to require that the sets be nonempty. The matrix M in Corollary 1 can be (i) singular, (ii) nonsingular but not sign-nonsingular, or (iii) sign-nonsingular. The first two cases correspond to equal unions.

Recently McCuaig showed that several related problems, including the recognition of sign-nonsingular matrices, could be solved in polynomial time [7]. Working independently, Robertson et al. also made this remarkable discovery [10]. An extended abstract by all four authors appears in [8].

Our main results are Theorems 3 and 4 which characterize the equal union property when the number of ones in the incidence matrix is $\geqslant \binom{n+1}{2}$.

2. Main results

For an $n \times n$ 0–1 square matrix $M = (m_{ij})$, let $\bar{M}$ denote the matrix obtained from M by replacing any 1 in row i column j with the symbol x_{ij}. We define

$$d(M) = \det(\bar{M}).$$

Note that $d(M)$ is a polynomial in the variables x_{ij}. The number of nonzero terms in $d(M)$ is the permanent of M. Each nonzero term has a coefficient of 1 or -1. The condition that $d(M) \not\equiv 0$ is equivalent to the corresponding sets having a system of distinct representatives.

Lemma 1. *Let M be a $n\times n$ 0–1 matrix having $d(M) \not\equiv 0$. Then M is sign-nonsingular if and only if all terms of $d(M)$ have the same sign.*

Proof. We may assume $n > 1$ since the case when $n = 1$ is obvious. If M is not sign-nonsingular, then there exists an $n \times n$ singular matrix B with the same sign pattern. This implies that there are positive numbers that can be substituted for the variables of $d(M)$ to make it zero. This would be impossible if all coefficients of the polynomial had the same sign. Conversely, assume $d(M)$ has both positive and negative terms. Let $x_{1j_1} \dots x_{nj_n}$ be one of the positive terms. Replace each x_{ij_i} with $n!$ and replace all other variables with $1/(n!)^n$. The positive terms must have value at least $(n!)^n$. Each negative term can have absolute value at most $1/(n!)$. There are at most $n!$ negative terms, so the absolute value of the negative terms is at most 1. Since $n > 1$, $n! > 1$. We have found positive numbers for which $d(M)$ is positive. Similarly, we can find positive numbers for which the polynomial is negative. By the continuity of the polynomial, there must be positive numbers for which the polynomial is zero. So M is not sign-nonsingular. □

Lemma 2. *Let $F = (S_1, \dots, S_n)$ be a sequence of subsets of $\{x_1, \dots, x_n\}$ and let $x_i \in S_i$ for $i = 1, \dots, n$. Suppose, for some $i \neq j$, that $\{x_i, x_j\} \subseteq S_i \cap S_j$. Then F has the equal union property.*

Proof. Let F have incidence matrix M. By Corollary 1 and Lemma 1 it suffices to show that $d(M)$ has both a positive and negative term. Since each $x_i \in S_i$, M has all 1's on its main diagonal. And so $d(M)$ contains the term $x_{11} \dots x_{nn}$. But we also must have $m_{ij} = m_{ji} = 1$ and so $d(M)$ also must contain the term

$$-x_{11} \dots x_{i-1i-1} x_{ij} x_{i+1i+1} \dots x_{ji} \dots x_{nn}. \qquad \square$$

Lemma 3. *Let F be a family of nonempty sets not having the equal union property. Then F has a system of distinct representatives.*

Proof. For each $k > 0$, the union of any subfamily of k sets must contain at least k elements, or else we could apply Corollary 2 to obtain equal unions. But this property, by Hall's Theorem, is equivalent to the existence of a system of distinct representatives [11]. □

Theorem 2. *Let $F = (S_1, \dots, S_n)$ be a sequence of nonempty subsets of an n-element set X, possessing a system of distinct representatives. Let S_1' be any set in which $S_1 \subseteq S_1' \subseteq X$, and define $F' = (S_1', S_2, \dots, S_n)$. Then if F has the equal union property, so does F'.*

Proof. Let M be the incidence matrix for F and let M' be the incidence matrix for F'. Since F has a system of distinct representatives, $d(M) \not\equiv 0$. If F has the equal union property, by Lemma 1 $d(M)$ must have nonzero terms of opposite sign. But $d(M')$ must also have these same terms, so M' is not sign-nonsingular. Thus F' has the equal union property. □

Let M_n denote the set of all $n \times n$ 0-1 matrices M having $d(M) \not\equiv 0$. For $P = [p_{ij}]$ and $Q = [q_{ij}]$ in M_n, define $P \leqslant Q$ if and only if for all i, j, $q_{ij} = 1$ whenever $p_{ij} = 1$. The minimal matrices under this partial order are the permutation matrices, and the unique maximal element is the matrix with all 1's. Theorem 2 says that if P is not sign-nonsingular, then so are all $Q \geqslant P$. The next result guarantees that a maximal chain will have roughly half its matrices not sign-nonsingular.

Theorem 3. *Let $F = (S_1, \ldots, S_n)$ be a sequence of nonempty subsets of an n-element set. If $\sum_{i=1}^{n} |S_i| > \binom{n+1}{2}$, then F has the equal union property.*

Proof. By contradiction assume not. Then by Lemma 3 the S_i's have a system of distinct representatives $x_i \in S_i$, $i = 1, \ldots, n$. Let $M = (m_{ij})$ be the incidence matrix. Our assumption on the sum of the cardinalities of the S_i's means that M has more than $n(n+1)/2$ ones. By Lemma 2, however, for each $i \neq j$, we must have either $m_{ij} = 0$ or $m_{ji} = 0$. Therefore, there are at least $\binom{n}{2}$ zeros. The number of ones can be at most $n^2 - n(n-1)/2 = n(n+1)/2$, a contradiction. □

The condition in Theorem 3 can produce an incidence matrix that is either nonsingular or singular:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

The example $\{1\}, \{1,2\}, \ldots, \{1,2,\ldots,\mathrm{n}\}$ shows that the bound $\binom{n+1}{2}$ in Theorem 3 is tight. We will now characterize those families which achieve $\binom{n+1}{2}$ but do not have the equal union property. The result is most easily stated in terms of 0–1 matrices. Define

$$T_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The matrix T_3 is the unique incidence matrix, up to row and column permutations, of a strongly connected tournament of order 3.

Theorem 4. *Let M be an $n \times n$ 0–1 matrix having exactly $\binom{n+1}{2}$ ones. Then M is sign-nonsingular if and only if there exist permutation matrices P and Q such that PMQ is a block upper triangular matrix whose blocks are either* [1] or T_3.

Proof. Assume PMQ has the above form for some P and Q. It is easy to see PMQ is sign-nonsingular if and only if M is sign-nonsingular. However the determinant of a block upper triangular matrix is the product of the determinants of its blocks [3, ex. 6, p. 293]. It follows that a block upper triangular matrix is sign-nonsingular if and only if each of its blocks is sign-nonsingular. One can see that T_3 is sign-nonsingular either by computing the symbolic determinant $d(T_3)$, or by observing that the corresponding family $\{\{1,3\},\{1,2\},\{2,3\}\}$ does not have the equal union property.

Conversely, assume M has exactly $\binom{n+1}{2}$ ones and is sign-nonsingular. By Lemma 3, the corresponding family has a system of distinct representatives, and so there exist permutation matrices P_1 and Q_1 for which P_1MQ_1 has ones along its main diagonal. There are $\binom{n}{2}$ ones off the main diagonal. For each $i \neq j$, we can not have a one in both positions ij and ji, as this would force equal unions by Lemma 2. Therefore the digraph of P_1MQ_1 is a tournament. Now consider the strong components of this digraph. They must be linearly ordered. Each strong component may be reguarded as a strongly connected tournament By a theorem of Moser [2] any strongly connected tournament has directed cycles of all orders $\geqslant 3$. In particular, if any strong component has order four or more it must have a four-cycle. This would preclude sign-nonsingularity. Therefore each component has size one or three (since there can also be no 2-cycles.) Renumbering the vertices in the digraph to respect the linear order among the strong components gives the desired form. □

Our Theorem 4 is the 0–1 analog of Theorem 2.0.2 in [1]. If negative entries are permitted, one can have $n-1$ additional nonzero entries and the diagonal blocks can be of arbitrary order.

3. Applications

If G is a graph and v a vertex in G, the *open neighborhood of* v, denoted $N(v)$, is the set of all vertices adjacent to v. The *closed neighborhood of* v, denoted $N[v]$, is $N(v) \cup \{v\}$. For U a set of vertices, $N(U)$ is defined as $\bigcup_{v\in U} N(v)$ and $N[U]$ is defined as $\bigcup_{v\in U} N[v]$. D. Rall and S. Hedetniemi observe [9] that

Theorem 5. *In any graph having at least one edge there exist disjoint vertex sets U and W such that $N[U]=N[W]$.*

Theorem 5 holds because the complement of any minimal dominating set is also a dominating set. We can also establish Theorem 5 using the ideas of equal unions. Let $F=(N[v_1],\dots,N[v_n])$ be the sequence of closed neighborhoods. We may regard each v_i as a representative for $N[v_i]$. Since G has at least one edge, this family satisfies the hypothesis of Lemma 2 and therefore has equal unions.

Throughout this paper, our basic assumption is that we have a nonempty family of nonempty sets. We allow two sets in the family to be equal, although this guarantees the equal union property. In a graph, it is possible that two distinct vertices have the same closed neighborhood, or even the same open neighborhood.

When we use open neighborhoods as our sets, a graph might not have equal unions (consider K_2). We now wish to characterize all graphs for which there exist nonempty disjoint sets U and W with $N(U) = N(W)$. Let G be a graph with vertices v_i, none of which are isolets. The incidence matrix M of the sets $N(v_i)$ is the adjacency matrix of G. If $\det(M) = 0$ then the open neighborhoods have the equal union property. So assume $\det(M) \neq 0$. If G has the equal union property then $d(M)$ has nonzero terms of opposite signs. Let

$$x_{1\pi(1)}x_{2\pi(2)}\ldots x_{n\pi(n)}$$

be one nonzero term. Here π is a permutation on $\{1,\ldots,n\}$. No point is fixed under π since M has zeros on the main diagonal. We write

$$\pi = \pi_1 \ldots \pi_k, \tag{2}$$

where the π_j are disjoint cycles. Cycles of even length are odd permutations and cycles of odd length are even permutations. Therefore the sign of π is determined by the number of even length cycles. The cycles in (2) each have length $\geqslant 2$, and correspond to either edges or cycles (of vertices) in G.

For purposes of this discussion let us consider an edge to be a two-cycle. Then a *cycle cover* is a disjoint set of cycles that covers all vertices. It is *even* if the number of even-length cycles is even. Otherwise it is *odd*. It follows that

Theorem 6. *Let G be a graph having no isolets and adjacency matrix M. There exist nonempty disjoint vertex sets U and W such that $N(U) = N(W)$ if and only if either* $\det(M) = 0$ *or G has both an even cycle cover and an odd cycle cover.*

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